Chapter 6

Dynamics and Action

6.1 Dynamics

Dynamics, as mentioned earlier, are the rules for finding the temporal evolution of a system. In Newtonian Physics, this set of rules was succinctly summed up in the rule: $\vec{f} = m\vec{a}$, see Section 6.1 on page 153. For a while, we will forget about light and fields and the dynamics of these complex systems and just describe simple point particles that move around freely in a simple space. We will find a new way to formulate the rules of dynamics that are more general but still produce the old $\vec{f} = m\vec{a}$ when it is appropriate. The advantage will be that the new rules will work in circumstances in which Newton's Laws were inappropriate or just did not make sense. With these new rules, we will also find a more powerful understanding of the concepts of symmetry and include systems such as fields all in a single dynamical principle. We will also be able to use this new procedure to form a more solid understanding of the ideas of energy and momentum. One complication will be that in order to formulate the rule, we will need ideas about kinetic and potential energy that we formulated earlier. Before we are done, these same ideas will take on a very different and more useful form. We will be able to understand why the massless photon has momentum but first we need to build the necessary background.

6.1.1 Background on Formulation of Action

It is usually not emphasized that the original formulation of Newton's Laws applied to only a very restricted set of circumstances. In Section 6.1 on page 153, Newton's Laws were described as dealing with the effects of one system on another with the assumption that all the parts of the bodies were

basically point objects that could move freely in space. This was fine when talking about the planets but, even for some of the simplest cases, these conditions do not hold.

Consider the problem of the motion of a blackboard eraser tossed into the air in the front of the lecture hall with a twisting spinning motion. Each part of the eraser is subjected to a huge array of forces. For convenience you can think of the parts of the eraser as the atoms but, even without an atomic hypothesis, all the following considerations still hold. Each part of the eraser is subject to the force of gravity and each part is subject to internal forces from the other parts of the eraser. First, there is an absurd number of parts and forces between the parts and between the parts and the world outside the eraser. We simplify this situation somewhat by assuming that the effect of gravity is the same throughout the eraser and thus reduce these many gravitational forces to a single force acting at one point at the mass weighted center of the body. This is a good approximation for the case of a small eraser in the near vicinity of the earth.

More subtly, we know that, as the eraser twists and spins, the different parts of the eraser will effect other parts. In fact, if the eraser was not a reasonably rigid body and held together by cohesive forces, in the spinning twisting motion, the parts would fly apart. Because the eraser is rigid, there are internal forces that act to hold the respective parts in a fixed relationship to each other. These forces are very complicated. They are in a very real sense unknowable; they are what they have to be to maintain the rigid configuration. These are called constraint forces. The eraser is not an exception. A car on the highway has a constraint force from the road called the normal force that is whatever it has to be to stop the car from falling into the road. Actually, with a little thought it becomes clear that almost all systems have constraints. The direct application of Newton's laws to systems that are constrained is wrong or impossible. There are an abundance of forces – too many to handle. Worse yet is the realization that many of them are, in fact, unknowable. The forces hold the eraser as it moves through space are whatever they have to be to maintain the positional relationship between the parts of the eraser. These are generally not known and thus cannot be inserted into a simple Newtonian framework.

In many special cases, fixes were developed that allowed the use of Newton's laws for motion in the presence of constraints and it was well known that this was a problem to both Newton and his immediate followers. The general problem of the motion of systems with algebraically described con-

straints was solved by Joseph-Louis Lagrange[1]. The procedure that he developed is the modern method for articulating the dynamics of any system and is the one that we will use.

6.1.2 Introduction to Action

The modern approach to dynamics is based on the use of an extremum principle like Fermat's least time theory of light. There is a physical quantity that is called the action. In some sense, this is an unfortunate name for this because we have used the word in another context, see Section 5.2.1 on page 125, and it has a connotation in the conventional usage. The action is a quantity that we will define in detail later but for now understand that is a quantity evaluated over a trajectory in space and time. Up until now, we have dealt with paths in space. Now, we deal with trajectories but the principles are the same. For instance, the Fermat principle of least time required the time of passage of the light over the entire path between two points in space. Here the action is evaluated for a trajectory on space-time between two events, an initial position and time and a final position and time. Generally, the object moves over the trajectory that has the least action. Obviously, I will need to back up a little to make this clear and to establish the terminology.

We describe the motion of anything as a connected set of events in space-time, a path in space-time called the trajectory of the particle. The events labeled by a place and a time and are the fundamental entities and a trajectory is a catalogue of the places and as time evolves where the object went. Of the infinity of trajectories that can connect two events, the naturally occurring trajectory will turn out to be the one that has the least action.

Consider a piece of chalk tossed up from my hand and returning to my hand some short time later. I am dealing with only one spatial dimension, up. The zero of up is at my hand. The motion of the chalk is a continuous series of events that start with the toss at a time selected to be the zero of time and returns to my hand at a later time T. In between, the chalk has occupied a set of places at specific times between zero and T. If you know the places for all times in that interval you have a trajectory. In Figure 6.1 on page 156, we show the trajectory in a space-time diagram.

Any trajectory is only one of several that have the same total time interval T and start and stop at the same height. Why did nature chose the

[1]Joseph-Louis Lagrange was born in France in 1736 and died in 1830. In 1788, he introduced a new approach to Newton's formulation of mechanics that is the basis for our modern interpretation in 1788.

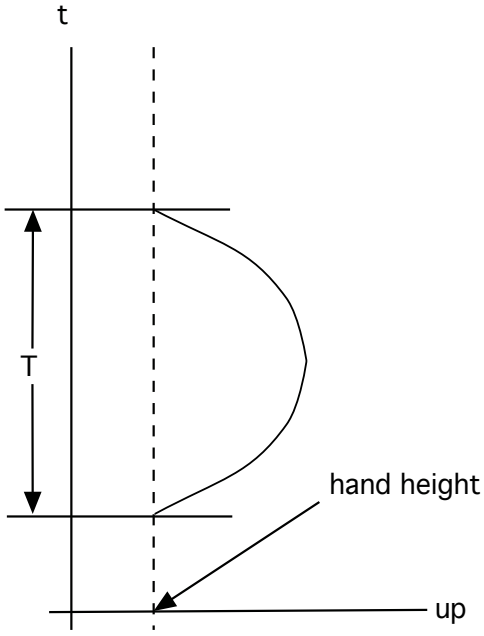

Figure 6.1: **Trajectory of a tossed piece of chalk** Chalk tossed from a height labeled zero rises with decreasing velocity until it reaches a peak and then returns to the hand after a time interval T.

one that she did? Severalpossible trajectories are shown in Figure 6.2 on page 157. It will turn out that our rule will be that nature chooses the trajectory from all the possible trajectories that has the least action. Since we have not yet defined the action, this is a little difficult to understand. Not only that but the approach is so different from the Newtonian that we do not have a developed intuition for this way of describing the chosen dynamic but some of the features of 'Least Time' articulation of paths for light carry over from our analysis of the properties of light, see Section 16 on Page 345. Like in the case of our 'Least Time' criteria for paths for light, this criteria is global and the minimizing property makes the 'Least Action' criteria produce a unique trajectory. This eliminates trajectories like the red one in Figure 6.2 on page 157 because the time flipped version has the same action. The issue of how high will require some analysis but we will find that action for this case is a combination of dependence of the maximum of the height of the trajectory above the hand height and the velocity it needs to get to the appropriate height.

If you were approaching this problem from the Newtonian point of view,

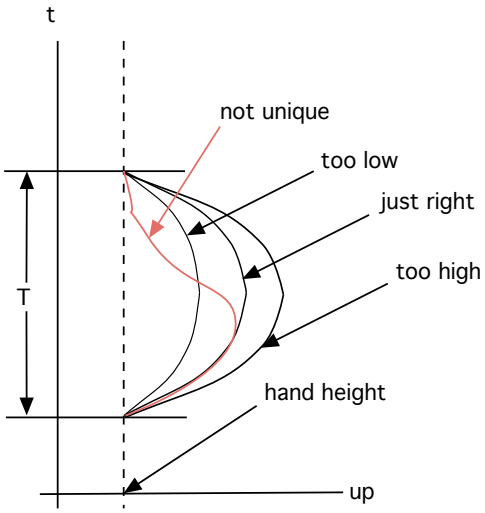

Figure 6.2: **Possible trajectories for a tossed piece of chalk** There are an infinity of trajectories that can connect the event at the start of the toss with the event at the return of the chalk to the hand at a later time T.

you would have used $\vec{f} = m\vec{a}$ and said that the chalk starts from a given place and given speed. Because there is a force, the attraction of the earth for the chalk, there is an acceleration. Since there is an acceleration, the velocity changes. The velocity changes until it is reversed at the maximum height and starts to fall. While all this is happening, the chalk is tracing out a smooth arc in space time. This description is very different than the one that we will be using for action. In the Newtonian formulation, the determination of the trajectory is done at each instant of time at the place at which the chalk is at that time. The action approach on the other hand deals with the action over the entire trajectory. This is a global approach to dynamics very similar to the 'Least Time' criteria of Fermat. It will be difficult to reconcile these disparate seeming approaches but you have to recover the Newtonian approach for the case in which the chalk can be treated as a point particle and free to move up and down without constraint. This will be covered in Section 6.1.5 on page 162

6.1.3 Definition of Action

Instead of $\vec{f} = m\vec{a}$ acting at each point on the body, there is now a new rule: minimize the action over the trajectory. In other words, nature chooses the least action trajectory from all the trajectories that share the same initial

and final event. This is a formulation of motion that is very much like that of Fermat's Least Time formulation for the paths of light in Section 3.2 on page 57. To determine the trajectory, you pick two events, an initial event, x_0 and t_0, and a final event, x_f and t_f. There is a quantity called the action that is computed for every segment of the trajectory. Choose all possible trajectories and thenatural trajectory is the one that has the least action.

The action is defined from a function of the positions and velocities called the Lagrangian. In this approach to dynamics, instead of trying to figure out what forces are causing the motion, you try to find what the correct Lagrangian is. In a real sense, when a modern physicist develops a new fundamental theory of some phenomena, it is by finding the correct Lagrangian so that the trajectory that yields the least action using that Lagrangian is the one that occurs naturally.

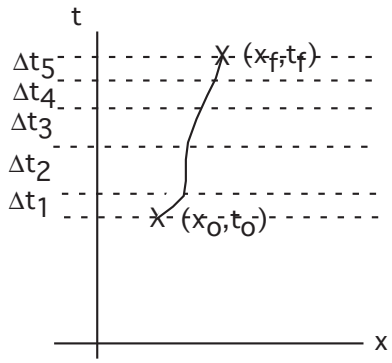

Figure 6.3: **Trajectory for the computation of the action** In order to compute the action for a given trajectory, the trajectory is divided into time slice pieces. For each time slice, the positions and the velocity can be determined. The action is then computed for that time slice and the contributions of each time slice are added to produce the overall action. The sizes of the time slices are determined by the rate of change along the trajectory.

There is a slight technical difference in this case and the case of Fermat's least time. In this case, we create our trajectory segments by creating time slices, see Figure 6.3 on page 158. For Fermat, the segments were sections along the length of the curve. As in the case of least time, the size of the time slices depends on the trajectory and the precision required. This gives a special role to the time variable. Also although we say all possible trajectories, for now, we will only deal with trajectories that advance in time

positively. We will be able to lift this condition later, Section **??** on page **??**.

For a simple point object like the piece of chalk moving up and down, the Lagrangian depends on the position and velocity of the object. Given the Lagrangian, the action is

$$S(x_f, t_f, x_0, t_0; trajectory) = \sum_{trajectory, x_0, t_0}^{x_f, t_f} (L(x(t), v(t))\Delta t \tag{6.1}$$

Action has the dimensions of an energy times a time. Although this makes the dimensions easy to remember, it is misleading. As we will learn later, the concept of energy is derivative from the action not the other way around, see Section 7.4 on page 194. It would be better to say that energy is dimensionally an action divided by a time. In terms of fundamental dimensional units, the units of action are $\frac{\text{mass}\times\text{length}^2}{\text{time}}$. From Equation 6.1 on page 159, the Lagrangian itself has the dimensions of an energy, $\frac{\text{mass}\times\text{length}^2}{\text{time}^2}$.

The rule that Lagrange found that would reproduce $\vec{f} = m\vec{a}$ for unconstrained systems and also work for more general situations is that the Lagrangian, $L(x(t), v(t))$, should be the difference in the kinetic energy and the potential energy.

$$L(x(t), v(t)) = \frac{mv^2}{2} - V(x) \tag{6.2}$$

where $V(x)$ is the potential energy. Later, Section 6.1.5 on page 162, we will show how this reproduces Newton's laws. It is important to again point out that although this approach requires that you know the kinetic energy and potential energy that these concepts are actually derived from the actions and not the other way. For now, it seems that you need to know the potential energy before you can write the Lagrangian. This is only for historical and pedagogical reasons. When a modern physicist is struggling with understanding some basic new phenomena, it is the other way around. We start with a Lagrangian and then see what the consequences are. It will also turn out that since the actions become the basis of all dynamics, it is the idea that theories that unify other earlier independent theories are considered unified when all the consequences of the theory arise from a single controlling Lagrangian. In modern language, Maxwell unified the electric and magnetic forces because the entire ensemble of equations is derivable from a single Lagrangian and the least action principle.

6.1.4 Trajectory of a Free Particle

To test our new dynamic, let's look at the simplest situation possible – a free particle. A free particle is one that has no forces acting on it. All places have the same energy value and thus $V(x) = C$, a constant, but if there is no preferred places and all places are the same then we may as well set the potential, the energy of a place, to zero. Using Lagrange's rule to get the solution for the free particle in old fashioned physics, we chose the Lagrangian that is just the kinetic energy or $L(v(t)) = \frac{mv^2}{2}$. To make it even simpler, let's require that the released particle is to return to the original position after a time T. The action is

$$S(0,0,0,T,traj.) \;=\; \sum_{traj.,0,0}^{0,T} \frac{mv^2}{2}\Delta t \tag{6.3}$$

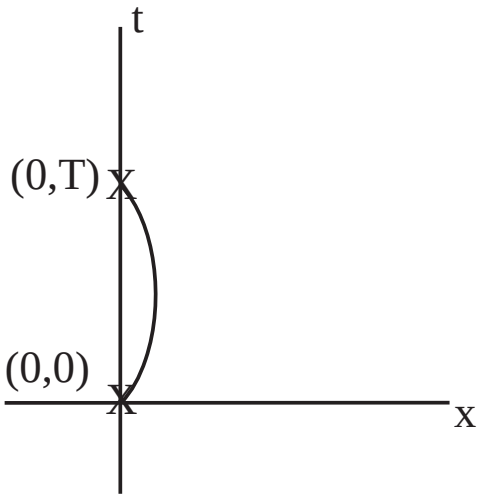

Figure 6.4: **Space-time diagrams for the action for a free particle** A particle with no forces acting on it moves between two events,$(0,0)$ and $(0,\mathrm{T})$. A possible trajectory is shown. Our experience with force free motion is that the straight line trajectory is the one that nature chooses; the particle remains at the point of release.

As was stated in the review section, Section 6.1 on page 153, a free particle at rest will remain at rest. Therefore, the natural trajectory for this case is the one that is at the starting place at all times. This is a straight

line along the t axis connecting $(0,0)$ and $(0,\mathrm{T})$. How do we obtain this same result using action?

Note that the action is a positive definite quantity for all velocities. Therefore any trajectory that has a non-zero velocity anywhere in the time interval will have a positive action. The trajectory that has $v(t) = 0$ for all t in the interval has an action of zero. This is clearly a minimum of the action since all other trajectories for these conditions will have a positive action. Thus this is the natural path. Actually any Lagrangian with v^2 in it will accomplish the same thing. The m is in it to give it the correct dimensions and the 2 for historical reasons. In fact, the m that is in the Lagrangian is the definition of mass. More on this later, see Section 15.3 on page 336.

Using this same result and remembering the material on Galilean invariance in Section 6.1 on page 153, we can solve a more general problem. Suppose we have a free particle that moves through the two events $(0,0)$ and (x_f, t_f). Again, since the particle is free, the natural trajectory is the straight line connecting these events. To an observer moving by us at a speed of $v = \frac{x_f}{t_f}$, the object is a rest during the entire time interval. To that observer it is free and the initial an final events are $(0,0)$ and $(0,t_f)$ and the natural path is the straight line along the t axis as before. Thus to us the natural trajectory will be the straight line with slope $\frac{t_f}{x_f}$. Let's obtain this same result with a direct analysis.

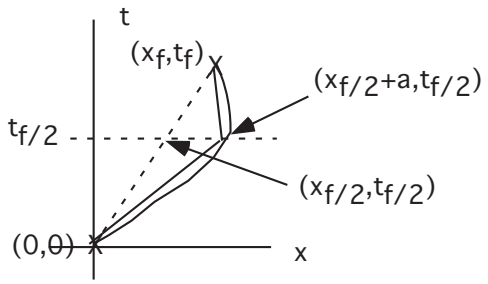

Figure 6.5: **Space-time diagrams for the action for a free particle that changes position** A particle with no forces acting on it moves between two events,$(0,0)$ and (x_f, t_f). A possible trajectory is shown. The general trajectory connecting these events would be very difficult to describe. We will approximate the trajectory with a trajectory that is kinked at the mid-time and straight otherwise.

Consider a general trajectory connecting events $(0,0)$ and (x_f, t_f), see Figure 6.5 on page 161. Our problem is to find all possible trajectories

between these events and then, for each trajectory, find the action. As we discussed about paths when dealing with the Fermat's least time approach to optics in Section 3.3.7 on page 76, path space is a rich mathematical structure. We want to do analysis. To do analysis we have to reduce the complexity of path space to something that can be described by functions. There are all these same difficulties when dealing with trajectories. To simplify our trajectory space, we reduce the trajectories that we consider to those that are "once kinked". Place the kink along the line $t = \frac{t_f}{2}$, see Figure 6.5 on page 161. In this reduced space, trajectories can be labeled by the distance, a, of the kink from the event $(\frac{x_f}{2}, \frac{t_f}{2})$ along that line. Using this trajectory in the appropriately modified Equation 6.3 on page 160 to take account of the new ending event, and the fact that the inverse slope of the line is the velocity in that segment, it is easy to compute the action for the trajectory labeled a. It is

$$S(0,0,x_f,t_f,traj=a) = \frac{m}{2}\left(\frac{\left(\frac{x_f}{2}+a\right)^2}{\frac{t_f}{2}} + \frac{\left(\frac{x_f}{2}-a\right)^2}{\frac{t_f}{2}}\right). \qquad (6.4)$$

This is an even function of a and thus has a minimum at $a = 0$. This confirms our result that the natural trajectory, the constant velocity trajectory, is the least action trajectory.

6.1.5 Proof that the Least Action Reproduces Newtonian Physics

See Feynman's famous lecture on action. It was handed out in class and is posted on Canvas.

For those who are intimidated by his vocabulary and technique, we will follow up with an approach that is hopefully more accessible. The major issue is the question of 'over all trajectories'. In Section 6.1.3 on page 157, we indicated that you could recover the usual statement of mechanics known as Newton's laws by defining the Lagrangian as the kinetic energy minus the potential energy. In this section, we will prove this. We will use the most simple and direct method for controlling the problem of the size of trajectory space by using a simple time sliced reduction of the trajectories to an $(\mathbb{R}^1 \times \mathbb{R}^1)^n$ by labeling the trajectory with the set $\{x_i, t_i\}$, where x_i is the position on the trajectory at the time slice t_i and n is the number of time slices and is a large number.

We are interested in all possible trajectories that connect an initial event (x_0, t_0) and a final event (x_f, t_f), see Figure 6.3 on page 158. Consider the

case of n time slices. Any trajectory that connects the initial and final events is labeled by the set of numbers $\{x_i, t_i\}$. We construct the action in the prescribed fashion, $L = m\frac{v^2}{2} - V(x)$. As in the earlier examples, the velocity in any segment, which is approximated by the straight line segment between two time slices, is the change in position in the segment divided by the time interval, $\frac{x_i - x_{i-1}}{t_i - t_{i-1}}$. Similarly, the position in the segment is the average position over that segment[2], $\frac{x_i + x_{i-1}}{2}$. Thus following the first of Equations 6.1 on page 159, the action for any trajectory is

$$S(x_f, t_f, x_0, t_0; \{x_i, t_i\}) = \sum_{seg.\,\{x_i,t_i\},x_0,t_0}^{x_f,t_f} \left(\frac{m}{2} \left(\frac{x_i - x_{i-1}}{t_i - t_{i-1}} \right)^2 - V\left(\frac{x_i + x_{i-1}}{2} \right) \right) (t_i - t_{i-1}), \quad (6.5)$$

where, as indicated, we sum over all the segments between (x_0, t_0) and (x_f, t_f).

Assuming that, from the very large set of possible $\{x_i, t_i\}$, there exists one that minimizes the action and is thus the naturally occurring trajectory. Designating this trajectory $\{\bar{x}_i, \bar{t}_i\}$, we can calculate the action with Equation 6.5 on page 163. We can also examine the nearby trajectory which is the same as $\{\bar{x}_i, \bar{t}_i\}$ everywhere except at some particular time slice for the natural trajectory, $\bar{t}_j$, and, for this new trajectory, the position coordinate differs from the naturally occurring one by a small amount. In other words, the new nearby trajectory is the one for which $\{x_i, t_i\} = \{\bar{x}_i, \bar{t}_i\}$ except when $i = j$ and, for that case, the position time pair is $(x_j, t_j) = (\bar{x}_j(1 + \epsilon), \bar{t}_j)$ where ϵ is a small number, $\epsilon \lll 1$. ϵ is such a small number that we can neglect all powers of ϵ greater than one. As we will see the ϵ^0 terms will drop out of the change in the action and thus we need only deal with the ϵ^1 term.

From Figure 6.6 on page 164, it is clear that the difference in the action of the naturally occurring trajectory and the nearby one requires only calculating the segment of the trajectories before and after the incremented one. This not only simplifies the calculation but also shows how this use of near in trajectory space can be used to convert a global statement of dynamics to one that is local in time like Newton's laws, see Section 6.1.1 on page 153.

[2]Depending in the nature of the potential function a differently weighted average may be more appropriate but if the time slices are small enough this simple average is adequate.

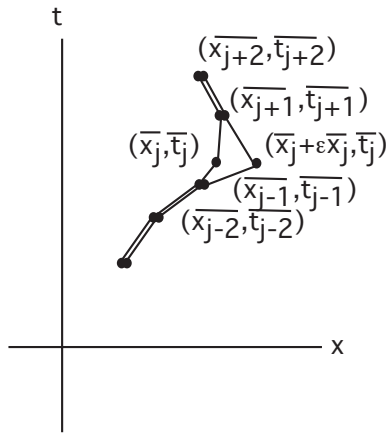

Figure 6.6: **Natural Trajectory and Nearby Trajectory** In the time sliced formulation of action, trajectories are labeled by the set $\{x_i, t_i\}$ where i ranges from 0 to n where n is the number of time slices. The naturally occurring trajectory is the one that minimizes the action and is designated by the $\{\bar{x}_i, \bar{t}_i\}$. The condition of minimum is established by requiring that the change in the action between the naturally occurring trajectory and any nearby trajectory vanish to first order in the measure of nearness. A measure of nearness for trajectories is to increment the space component of a trajectory at some time slice by a small amount. Above the natural trajectory which has many time slices is incremented at the j th slice to produce the nearby trajectory. In the calculation of the difference of the action over the naturally occurring and nearby trajectories only the parts of the trajectories around the j th slice are relevant. This converts a global condition for the naturally occurring one to one that is local in time like the Newtonian statement of mechanics.

Writing the difference in the actions for the naturally occurring trajectory and the nearby one,

$$\begin{aligned}
\delta S &= S_{\text{near by}} - S_{\text{naturally occurring}} \\
&= \left\{ \frac{m}{2} \left(\frac{\bar{x_j}(1+\epsilon) - \bar{x}_{j-1}}{\bar{t_j} - \bar{t}_{j-1}} \right)^2 - V\left(\frac{\bar{x}_j(1+\epsilon) + \bar{x}_{j-1}}{2} \right) \right\} (\bar{t_j} - \bar{t}_{j-1}) \\
&\quad + \left\{ \frac{m}{2} \left(\frac{\bar{x}_{j+1} - \bar{x}_j(1+\epsilon)}{\bar{t}_{j+1} - \bar{t}_j} \right)^2 - V\left(\frac{\bar{x}_{j+1} + \bar{x}_j(1+\epsilon)}{2} \right) \right\} (\bar{t}_{j+1} - \bar{t}_j) \\
&\quad - \left\{ \frac{m}{2} \left(\frac{\bar{x_j} - \bar{x}_{j-1}}{\bar{t_j} - \bar{t}_{j-1}} \right)^2 - V\left(\frac{\bar{x}_j + \bar{x}_{j-1}}{2} \right) \right\} (\bar{t_j} - \bar{t}_{j-1}) \\
&\quad - \left\{ \frac{m}{2} \left(\frac{\bar{x}_{j+1} - \bar{x}_j}{\bar{t}_{j+1} - \bar{t}_j} \right)^2 - V\left(\frac{\bar{x}_{j+1} + \bar{x}_j}{2} \right) \right\} (\bar{t}_{j+1} - \bar{t}_j) \,. \qquad (6.6)
\end{aligned}$$

In order to proceed, we will use Newton's approximation, $f(x+\epsilon) \approx f(x) + \epsilon \frac{df}{dx}(x)$ which is valid for any small ϵ and any smooth $f(x)$ for the potential function in the near by trajectory terms. Expanding and dropping terms of order ϵ^2,

$$\begin{aligned}
\delta S &= \epsilon x_j \left\{ m \left\{ \frac{\bar{x}_j - \bar{x}_{j-1}}{\bar{t}_j - \bar{t}_{j-1}} - \frac{\bar{x}_{j+1} - \bar{x}_j}{\bar{t}_{j+1} - \bar{t}_j} \right\} \right. \\
&\quad - \frac{1}{2} \left\{ \frac{dV}{dx} \left(\frac{\bar{x}_j + \bar{x}_{j-1}}{2} \right) (\bar{t}_j - \bar{t}_{j-1}) \right. \\
&\quad \left. \left. + \frac{dV}{dx} \left(\frac{\bar{x}_{j+1} + \bar{x}_j}{2} \right) (\bar{t}_{j+1} - \bar{t}_j) \right\} \right\} \qquad (6.7)
\end{aligned}$$

This result can best be interpreted by first multiplying through by $\frac{\bar{t}_{j+1} - \bar{t}_{j-1}}{\bar{t}_{j+1} - \bar{t}_{j-1}}$, and reorganizing

$$\begin{aligned}
\delta S &= \epsilon x_j \, (\bar{t}_{j+1} - \bar{t}_{j-1}) \left[-m \frac{1}{(\bar{t}_{j+1} - \bar{t}_{j-1})} \left\{ \frac{\bar{x}_{j+1} - \bar{x}_j}{\bar{t}_{j+1} - \bar{t}_j} - \frac{\bar{x}_j - \bar{x}_{j-1}}{\bar{t}_j - \bar{t}_{j-1}} \right\} \right. \\
&\quad - \frac{1}{2} \left\{ \frac{dV}{dx} \left(\frac{\bar{x}_j + \bar{x}_{j-1}}{2} \right) \left(\frac{\bar{t}_j - \bar{t}_{j-1}}{\bar{t}_{j+1} - \bar{t}_{j-1}} \right) \right. \\
&\quad \left. \left. + \frac{dV}{dx} \left(\frac{\bar{x}_{j+1} + \bar{x}_j}{2} \right) \left(\frac{\bar{t}_{j+1} - \bar{t}_j}{\bar{t}_{j+1} - \bar{t}_{j-1}} \right) \right\} \right] . \qquad (6.8)
\end{aligned}$$

For $\delta S = 0$, since all the factors in front of the square bracket are non-zero, the terms inside the square bracket must vanish at each t_i. This is a condition on the naturally occurring trajectory. The terms in the curly

brackets on the first line of Equation 6.8 are the change in velocity in the two relevant segments of the trajectory which when divided by the time interval over the two segments is the acceleration at t_j. The mass times this acceleration must equal the remaining terms in the curly bracket. These are position and time segment weighted averages of the quantity $\frac{dV}{dx}$ evaluated in the two relevant segments. Since in Newtonian mechanics $F = -\frac{dV}{dx}$, we have that for each i along the naturally occurring trajectory

$$m\bar{a}_i = -\frac{dV}{dx}(\bar{x}_i) = F.$$

6.2 Examples of Actions

6.2.1 Gravitation near a flat earth

As a simple example that we are all familiar with, consider the case of motion above the surface of the earth. Here the energy of position, the potential energy, is due to the gravitational interaction of a massive body with the earth. For this case, the potential energy at a height h above the earth is $V(\vec{r}) = -\frac{Gm_e m}{R_e+h}$, where m_e is the mass of the earth, m the mass of the body, and R_e is the radius of the earth. For motion near the surface, a few meters up or down, from "Things Everyone Should Know," Section 1.4.2 on page 16, we can use $(1+x)^n \approx 1 + n\ x$ for $x \ll 1$ to reduce this to

$$V(h) = -m\left(\frac{Gm_e}{R_e}\left(1 - \frac{h}{R_e}\right)\right) = V(R_e) + mgh,$$

where we recognize $g = \frac{Gm_e}{R_e^2}$. Since this potential is to be used in an action, as we will see later in Section 7.4 on page 194, changing the action by a constant does not change the physical results in a significant way, we can drop the $V(R_e)$ term. This reduces the potential energy for objects moving in the near vicinity of the earth to

$$V(h) = mgh. \tag{6.9}$$

Another way to look at this result is to say that for motion restricted to be near the surface of the earth, the earth appears as an infinite plane. In this case, the force of gravity above the plane can not depend on anything, in particular, the height above the plane or the position sideways over the plane. Thus the force also can only be toward or away from the plane. Then realizing from the analysis above in Section 6.1.5 on page 162 that the

change in potential as you change position is the force, the only form for the potential in this case is $mgh + \text{constant}$.

For now, let us continue to consider only up and down motion, not any sideways motion. The potential energy is mgh where h is the height. Thus the action for any trajectory between an initial height, h_0 at time t_0 and final height, h_f at time t_f is

$$S(h_0, t_0, h_f, t_f; traj.) = \sum_{traj., h_0, t_0}^{h_f, t_f} \left(\frac{mv^2}{2} - mgh \right) \Delta t \tag{6.10}$$

where the path is given by $h(t)$. Note that if you know $h(t)$, you also know $v(t)$. You can see from the form of the action that you will lower the action by having $h(t)$ to be at large h for as much time as possible. The problem is that since the initial and final position and time are given, it takes high velocity to get to large h. The high velocity increases the action $\Rightarrow$ There is a compromise between the negative contributions by going too high and the velocity needed to get there. This is the trajectory that the particle follows naturally, see Figure 6.2 on page 157.

Let's get more specific. This is again the problem of a piece of chalk tossed up in the air. First the simplest case, the chalk is released and returns to the same height after a t time T.

We need to study the action for all trajectories connecting these events. Again, because of the complexity of the idea of all trajectories, we will need to reduce the number of trajectories. A first step is to use our experience to limit ourselves to simple trajectories that rise smoothly to a peak at some height a at which time the velocity is zero and then returns over a trajectory that is a reflection of the one on the rise. Our natural trajectory must be in that family. This is still a very rich family and too rich to do analysis. This is the same problem that we had with the Fermat's Least Time, Section 3.3.7 on page 76, and the free particle, Section 6.1.4 on page 160. As in the latter case, the once kinked path can be used to approximate the family of smooth trajectories that have these properties, see Figure 6.7 on page 168. Here again the variable a is the height of the approximate trajectory but more importantly now it is a label that can be used to specify the particular trajectory from the family with which we are dealing. We are labeling all possible trajectories with a single parameter–putting it on an $\mathbf{R^1}$.

Since this approximate trajectory is broken line segments, it is relatively

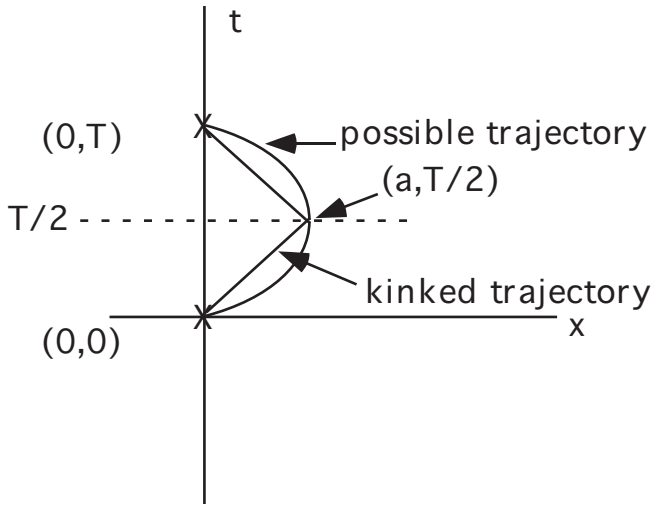

Figure 6.7: **Possible trajectory for the action for a particle in a uniform gravitational field** A piece of chalk is tossed upward and caught later at the the same height. A possible trajectory is shown. The natural trajectory is one from the family of smooth trajectories that rise to a peak at a height a smoothly and then return to a lower height on a reflected trajectory. This is still a large family of trajectories. We can approximate the members of this family with a once kinked trajectory with the same height at the time $\frac{T}{2}$.

easy to compute the action.

$$S(0,0,0,T;traj.) = \sum_{(0,0)\ \text{traj.}}^{(0,T)} \left(\frac{mv^2}{2} - mgh\right)\Delta t. \tag{6.11}$$

For a straight line path, v is a constant and is the inverse slope of the line, and is $\frac{a}{\frac{T}{2}}$ in magnitude for both segments. The height is a more subtle question since it varies with time from 0 to a. Being reasonable, we can use the average height, $\frac{a}{2}$. For the sophisticates among you, there is the problem that the concept of average is a not trivial, see Section 6.2.6 on page 178. Thus the action for the first segment is

$$S_1(T,a) = \frac{ma^2}{2\left(\frac{T}{2}\right)^2}\frac{T}{2} - \frac{mga}{2}\frac{T}{2}. \tag{6.12}$$

Note that once I have made a mapping of the paths onto the line that S

becomes a regular function of the path label, a, instead of a functional. Although the velocity is negative, since only v^2 enters the lagrangian, the action on the second segment is the same and the total action is

$$S(T,a) = 2S_1(T,a) = ma\left(2\,\frac{a}{T} - \frac{g}{2}T\right) \tag{6.13}$$

This has zero's at $a = 0$ and $a = \frac{gT^2}{4}$. The dependence of the action on the path label a is shown in Figure 6.8 on page 169. I have used dimensions in which $g = T = 1$.

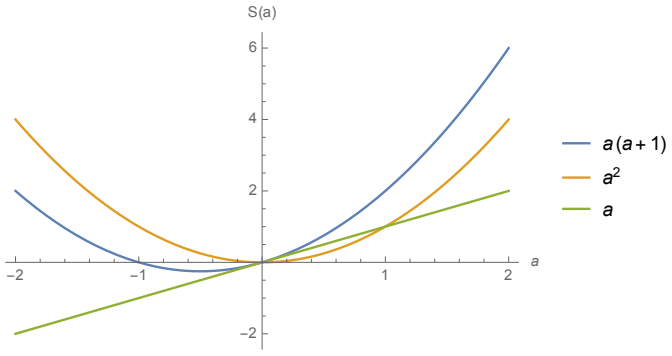

Figure 6.8: **Action as a function of** a The action as a function of of the trajectory label a. This curve is a combination of a parabola, $\frac{mT^3}{6}a^2$, concave up with its vertex at the origin and a straight line, $\frac{mT^3}{6}a$, with positive slope through the origin. The units are $m = \frac{1}{6}$, $T = 1$. It is important to realize that for any polynomial adding a to it of lesser degree only shifts the plot and that in this case the two polynomials share a common zero so this point does not shift.

We can see that there is a minimum half way between the two zero's at $a = 0$ and $a = \frac{gT^2}{4}$. This implies that the trajectory from this set in this approximation that is the least action trajectory is the one with

$$a_{least\ action} = \frac{gT^2}{8}. \tag{6.14}$$

Since this is not only the path selecting parameter but is also the height of the trajectory. We get that the height is $\frac{gT^2}{8}$.

6.2.2 Same Example done another way*

I am going to do some mathematics here that I do not expect that you will be able to reproduce. I do this to show you that it can be done and that the

ideas of mathematics are useful. You are not expected to do integrals and take derivatives although you should be able to follow a development using them.

Once again, we want to examine the case of an object of mass m moving in the vicinity of the earth. We can also guess that the correct answer for the height as a function of time is a parabola, all parabolas that fit the time interval are of the form $h(t) = at(t-T) \Rightarrow v(t) = 2at - aT$, where a is label of the path in path space[3]. In this case, a has the dimension of an acceleration, $L \stackrel{\text{dim}}{=} a \times T^2$ or $a \stackrel{\text{dim}}{=} \frac{L}{T^2}$.

The Lagrangian is $L = \frac{1}{2}mv^2 - mgh$ and the action is

$$\begin{aligned} S(a) &= \int_{(x_0,t_0),\text{Path}}^{(x_f,t_f)} \Big(\frac{1}{2}mv^2 - mgh\Big)dt \\ &= m\int_0^T \Big(\frac{1}{2}(2at - aT)^2 \ - gat(t-T)\Big)\,dt \\ &= m\Big(\frac{a^2T^3}{6} + \frac{1}{6}agT^3\Big) \\ &= \frac{mT^3}{6}a\,(a+g)\,. \end{aligned} \tag{6.15}$$

To find the minimum, we can again realize that there are two zeros of S(a). One at $a = 0$ and one at $a = -g$. The minimum is half way between them at $a_{least\ action} = -\frac{g}{2}$

Otherwise and since S is now a function, we can take the derivative of S(a) with respect to a and set it equal to zero. Thus

$$\begin{aligned} \frac{dS}{da} &= \frac{d}{da}\left(\frac{mT^3}{6}a(a+g)\right) \\ &= \frac{mT^3}{6}\left\{a + (a+g)\right\} \\ &= \frac{mT^3}{6}\left(2\,a + g\right) \end{aligned}$$

or $a_{least\ action} = -\frac{g}{2}$ is the natural trajectory. . Another important feature of this situation is that $\frac{d^2S}{da^2}(a) = \frac{mT^3}{3} \geq 0$. This guarantees that the minimum is a true local minimum. This is also shown by the fact that $S(a)$ is a parabola that is concave upward. Figure 6.9 on page 171, shows how the action varies with a. Again I have used units with $g = T = 1$.

[3]Note that since the path label a is an element of an $\mathbf{R}^1$ the action is now a function of the path label and I am now able to indicate S(a) instead of the functional indication $S[a]$.

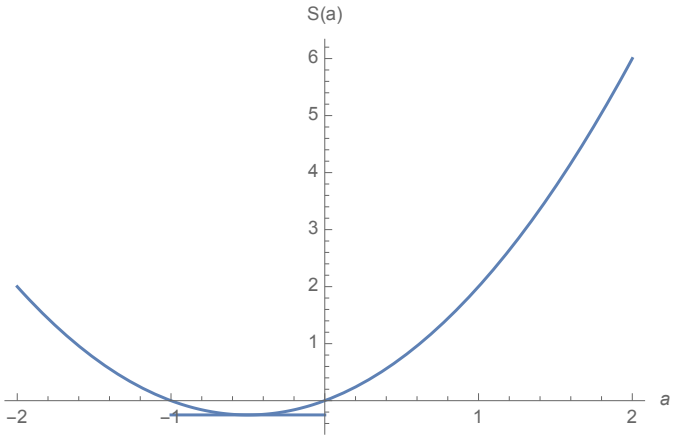

Figure 6.9: **Action as a function of a as an acceleration** Action as a function of a when the parameter a has the dimensions of an acceleration and the units of g and m are both set equal to 1. This example shows that the trajectory label does not have to be a height. You can use any convenient $\mathbf{R}^1$.

6.2.3 Simple Scattering of Two Particles

Two particles, one of mass m_1 and the other of mass m_2 collide. After the collision, the particles move away from each other, both still with masses m_1 and m_2. This is a very special problem whose importance cannot be over emphasized. In a very real sense, when we probe the nature of the elementary constituents of matter, scattering experiments are the primary source of our knowledge. In addition, the process is so basic that it will allow us to begin to better understand many fundamental issues. For this subsection, we will make the simplest description[4]

How do we handle this process? First, we have to decide what is meant by two independent particles. Before the particles make contact, they move as if the other particle was not present, i. e. they are independent. It is reasonable therefore to assume that while they are apart or not interacting, the two particles actions add and are the usual free particle action. In other words, there is a free particle action the tells you all the properties of what is meant by a particle and its nature. For our construction of the action

[4]For now, we discuss the simplest construction of a scattering process. The subtlety will be elaborated when we have developed more tools for their articulation later in this section. Also see Section 15.1 on page 331, Section 7.4.4 on 199. For now we are pretending that we live in a pipe with an unstructured objects.

of the free particle in Section 6.1.4 on page 160, we used the Lagrangian $L(x, v) = \frac{mv^2}{2}$. The Lagrangian says the the object identified as a free particle does not treat different places differently and thus there is no x dependence in the Lagrangian. If we want to recover Newton's Law, see Section 6.1.5 on page 162, we use the usual classical kinetic energy. We will find that in other circumstances, that a different free particle Lagrangian is appropriate. If we wanted to describe something more complicated than a point particle, say a small rod, we would need elements that deal with what a rod is such as moment of inertia and directional variables.

By using as the action the sum of the single particle actions, the properties of the total system will be the sum of the properties of the parts. If we did this though, and this was the end of it, nothing interesting would ever happen; the particles would merely pass through each other unchanged in their motion. We want them to scatter. Thus in addition, we need to add a part that carries the interaction. The interaction will have a Lagrangian that is made up of relationship variables such as their separation in addition to the particle labels. In other words, the action is made up of the following parts:

$$\begin{aligned} \text{Total Action} &= \text{Free Action(variables particle 1)} \\ &+ \text{Free Action(variables particle 2)} \\ &+ \text{Interaction Action(variables particle 1,} \\ &\quad \text{variables particle 2, relationship variables).} \end{aligned} \tag{6.16}$$

Of course, it is actually redundant to list the relationship variables in the interaction action since they will be composed of the relationship variables of particle 1 and 2 anyway. The importance of displaying the relationship variables separately is to be able to say that, for a scattering situation, the interaction action is zero when the relationship variables such as the separation are large. In a collision, we assume that most of the time the particles travel toward or away from each other and that the interaction terms contribute only for a short time when the particles are in contact and thus this interaction term is small and does not add significantly to the total action of the process. Another point to note is that, since the interaction terms are dominated by the relationship variables, the contribution from the interaction action should be independent of where and when the collision takes place. Thus, we can write the action for this simple one dimensional

scattering process as

$$S = \sum_{(x_{10},t_{10}),Path}^{(x_{1f},t_{1f})} m_1 \frac{{v_1}^2}{2}\Delta t + \sum_{(x_{20},t_{20}),Path}^{(x_{2f},t_{2f})} m_2 \frac{{v_2}^2}{2}\Delta t + A, \tag{6.17}$$

where A represents the contribution of the action when the particles are in contact. The scattering process is shown in Figure 6.10 on page 173.

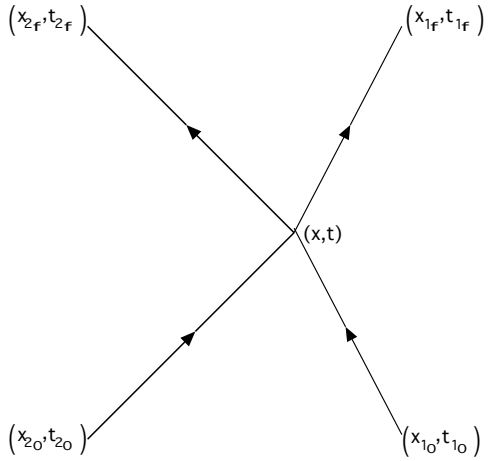

Figure 6.10: **Space-time diagram for a scattering event** Two particles of mass m_1 and m_2 free to move in one spatial dimension are directed at each other and collide at the event (x, t) and then move apart . A space-time diagram for a scattering event with particle one starting at event (x_{1_0}, t_{1_0}) and returning to (x_{1_f}, t_{1_f}) and particle two starting at event (x_{2_0}, t_{2_0}) and returning to (x_{2_f}, t_{2_f}) is shown. Although all trajectories connecting the initial and final events and the collision event should be examined, we know that free particles have a natural trajectory that is a straight line, see Section 6.1.4 on page 160.

We want to do all paths but we know that the straight path is the least action for a free particle and so all we need to do is use straight paths between the initial and collision and collision and final events. We can immediately write down the action as a function of the position and time of the collision. The coordinates of that event are the only free parameters in the problem.

Note that we are being consistent in our use of action. When you talk about collisions in the general physics class you set the initial velocities. Here we use the initial and final events. Evaluating the free particle actions,

for this system of trajectories, the action is

$$S = \frac{m_1}{2}\frac{(x - x_{1_0})^2}{(t - t_{1_0})} + \frac{m_2}{2}\frac{(x - x_{2_0})^2}{(t - t_{2_0})} + \frac{m_1}{2}\frac{(x_{1_f} - x)^2}{(t_{1_f} - t)} + \frac{m_2}{2}\frac{(x_{2_f} - x)^2}{(t_{2_f} - t)} + A. \tag{6.18}$$

We want to find the trajectory that has the least action and since we have now reduced the world of trajectories to the label of the collision point, x and t. The only term that we do not know is the contact term A. It could depend on any number of things; the hardness, compressibility and location in space and time of the collision. For our case, we assume that it is independent of every measure that we can impose; the particles are hard elastic spheres.

Thus we need to minimize this in what are now the labels, x and t. We can eliminate one of the steps by taking advantage of the requirements for uniqueness. All cases except those on the line $t = \frac{t_{1_f} - t_{1_0}}{2} = \frac{t_{2_f} - t_{2_0}?}{2}$ have partners with the same action. You could plot this and find the minimum by hand, see Figure 6.11 on page 175, but, if you allow me to use calculus, I can find a simple analytic expression for the $x = x_{min}$ and $t = t_{min}$ that yields the least action. This means taking the derivatives with respect to x and t and finding the value of x and t that satisfy $\frac{\partial S}{\partial x} = 0$ and $\frac{\partial S}{\partial t} = 0$. This x and t label the naturally occurring trajectory.

Take my word for it. The condition for a minimum in x is

$$m_1\frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2\frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} - m_1\frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} - m_2\frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} = 0 \tag{6.19}$$

or

$$m_1\frac{(x_{min} - x_{1_0})}{(t_{min} - t_{1_0})} + m_2\frac{(x_{min} - x_{2_0})}{(t_{min} - t_{2_0})} = m_1\frac{(x_{1_f} - x_{min})}{(t_{1_f} - t_{min})} + m_2\frac{(x_{2_f} - x_{min})}{(t_{2_f} - t_{min})} \tag{6.20}$$

Realizing that momentum is mv in classical physics and that v is the difference in positions divided by the the differences in times, this is the statement that the momentum into the collision is equal to the momentum out of the collision.

The condition that there is a minimum in t gives

$$\frac{m_1}{2}\frac{(x_{min} - x_{1_0})^2}{(t_{min} - t_{1_0})^2} + \frac{m_2}{2}\frac{(x_{min} - x_{2_0})^2}{(t_{min} - t_{2_0})^2} - \frac{m_1}{2}\frac{(x_{1_f} - x_{min})^2}{(t_{1_f} - t_{min})^2} - \frac{m_2}{2}\frac{(x_{2_f} - x_{min})^2}{(t_{2_f} - t_{min})^2} = 0 \tag{6.21}$$

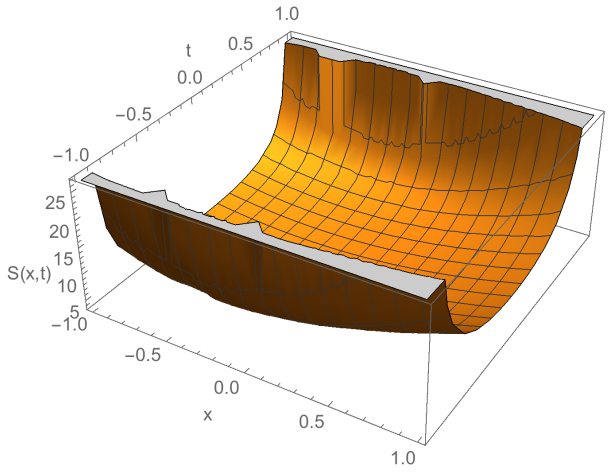

Figure 6.11: **Action for a Scattering Event** Action as a function of x and t for a scattering event shown in Figure 6.10 on page 173. There is a clear minimum and it occurs at the points at which Equation 6.20 on page 174 and Equation 6.22 on page 175 are satisfied.

or

$$\frac{m_1}{2}\frac{(x_{min}-x_{1_0})^2}{(t_{min}-t_{1_0})^2}+\frac{m_2}{2}\frac{(x_{min}-x_{2_0})^2}{(t_{min}-t_{2_0})^2}=\frac{m_1}{2}\frac{(x_{1_f}-x_{min})^2}{(t_{1_f}-t_{min})^2}+\frac{m_2}{2}\frac{(x_{2_f}-x_{min})^2}{(t_{2_f}-t_{min})^2} \tag{6.22}$$

Which is the same as the statement that the energy into the collision event is equal to the energy out of it.

Figure 6.11 on page 175 shows the action as a function of the position and time of the collision event. This is for the case that $\frac{m_2}{m_1}$ is 2 and the original and final events for particle 1 are $x_{1_0} = 1$, $t_{1_0} = -1$ and $x_{1_f} = 1$, $t_{1_f} = 1$ and for particle 2 are $x_{2_0} = -1$, $t_{2_0} = -1$ and $x_{2_f} = -1$, $t_{2_f} = 1$.

This exercise also gives us an interesting insight on what mass is. In an early assignment in this course, you were asked to devise a method for measuring mass that does not rely on gravity. Some of you came up with the idea of using collisions to define a mass scale. You can see that this analysis is directly relevant to that kind of definition. In the construction of the action, for the case of the single particle, mass is an overall factor; it is the thing you put in front of the v^2, in the action. If the world consisted of only one particle, mass would be irrelevant since all it does is multiply the action[5]. The process of finding the natural trajectory is unchanged by the

[5]At least the concept of inertial mass. The gravitational mass which generates the

an overall scale factor on the action. Mass becomes interesting only when you have more than one particle. If there is more than one particle, you can not remove all the masses with a single scaling factor. The ratios of the mass remain. Consider a scattering event between two particles with the initial and final positions of the two particles the same before and after the collision. If the particles had equal masses, the position of the collision event is at the center. The trajectories of both particles are equally kinked. On the other hand, the higher the mass ratio of say the second particle, the less the trajectory associated with that particle will kink when it collides with another particle. In the limit of a very large mass second particle, there is no bending of the second trajectory and it looks like the first particle has hit a brick wall. This is the essence of inertia.

6.2.4 A Small Sphere Hanging from a Fixed Point

Let's use the results of the previous analysis of the up-down motion over a flat earth to look at a more complex system, the simple pendulum. An example is a rather massive small sphere called a bob connected by a flexible light cord with one end attache to a fixed point in space above a flat earth. To even start on this problem we exercise our skills in 'Spherical Cowing' , see discussion on page 30. The cord is flexible and massless and the bob can be treated as a point mass. We will look at it's motion for small angular displacements from equilibrium, see Figure 6.12 on page 177.

To construct an action for a classical situation our first problem is to identify the kinetic energy. We have only one mass and as usual the kinetic energy is $KE = m\frac{(\vec{v})^2}{2} = m\frac{v_x^2}{2} + m\frac{v_x^2}{2} + m\frac{v_z^2}{2}$ where the velocity is measured in the three space over the flat earth with z going up down and x-y forming the plane of the surface of the flat earth. One complication is that the velocity is a vector quantity in a three dimensional space and our mass is constrained to move in a curved two dimensional space; see Figure 6.12 on page 177. If the angle θ is small, the v_z is small compared to v_y and v_y and can be neglected[6] of the mass's movement in three space with the motion on the projected plane of the flat earth.The potential energy is the same as in our previous case, $PE = mgh$. Putting a coordinate system on the flat

gravitational force is a totally different concept. The equivalence of these independent mass definitions is a foundational issue in the modern theory of gravity, see Chapter 16 on page 345.

[6]This is an aspect of "spherical cowing". The velocity $\vec{v}$ is constrained to be tangent to the places on the cupped surface. Therefore, at any place on the surface, the z component of the tangent is small compared to the components tangent.

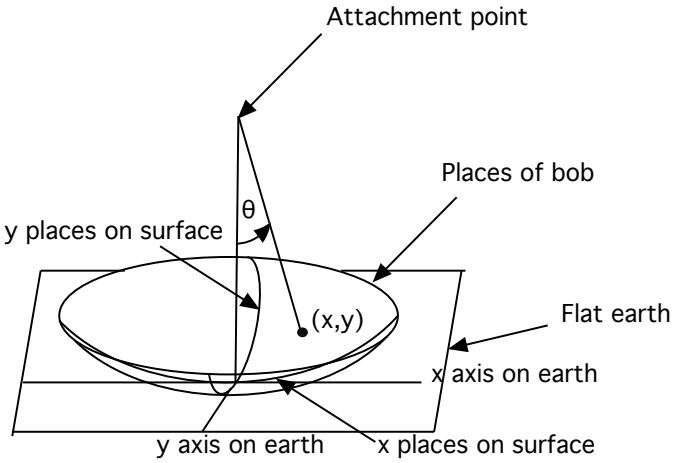

Figure 6.12: **The Simple Pendulum** A point mass, a bob, of mass m is suspended from a point over a flat earth at height h with a cord of length $L < h$. The mass is given a small angular displacement from the vertical and allowed to move freely between the place (x_0, y_0) and (x_f, y_f) in a time T.

earth with the z along the vertical and an xy plane on the earth surface. The orientation of the x axis is arbitrary since the system is rotationally symmetric for rotations about a vertical axis through the fastening point.

6.2.5 Action for the Stretched String

Although it is an unrealistic representation of a simple field, the idealized stretched elastic string is a wonderful example of a simple mechanical system which manifests field properties, see Section 5.3 on page 128. We developed a dynamic for this field which produced a broad range of motions including travelers. Since one of our principle purposes is to use the general concepts of action based physics to explain all physical phenomena our earlier discussion of action has to be extended to field phenomena. In this section we will articulate the procedures which are required to bring extended systems into an action formulation of mechanics via the example of the stretched string. We will follow the general protocols set forth by Lagrange that we used successfully for simple point masses but applied to the extended elastic system. This will provide the framework to developing the action formulation appropriate to the generic field systems.

Our basic model of the string is a connected set of point mass placed at

equal intervals each one connected to a nearest neighbor by a small elastic member, see Figure 5.1 on page 129. Following Lagrange's original prescription, the Lagrangian contains two parts; a kinetic energy component and a potential energy which, in this case, is the tension in the string times the stretch of the string associated with the configuration of the string.

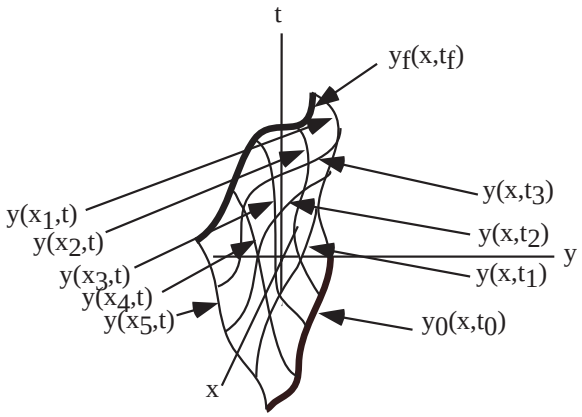

Figure 6.13: **String Fields for Action** In contrast to the case of the point particle, Figure 6.3 on page 158, the trajectory for the field between the initial field configuration and final field configuration is a two surface in a three space, two spatial directions and one time. The two spatial directions are the label for the places on the string, x, and one is the direction in which the string moves, y, assumed transverse. The naturally occurring field configuration is the surface configuration that minimizes the action. The dynamic for the field is found by looking at small variations about the naturally occurring configuration and requiring that the changes in the action vanish.

6.2.6 Digression on averages and slicing*

It should come as no surprise that most people do not think hard about what they mean by averages. This is often exemplified by the puzzle:

> Consider two towns that are one hundred miles apart, for instance Austin and College Station. You want to travel between them with an average speed of fifty miles per hour. You leave Austin but get caught behind a very long funeral procession that you cannot pass that is also going to Hicksville, half way between

> Austin and College Station. If the funeral procession held your speed to an average of twenty five miles per hour between Austin and Hicksville, how fast do you have to drive in the remainder of the trip to obtain your desired average of fifty miles per hour?

The accepted answer is that you have to go infinitely fast. This is because in the portion of the trip between Austin and Hicksville has taken two hours and, in order to average fifty miles per hour on a one hundred mile trip, you need two hours of travel time. Your time is all used up. Another answer that is often given is seventy five miles per hour in the second segment of the trip. Although not the accepted answer, there is a sense in which this answer is also correct.

How can there be two correct and different answers to the same question? The answer is that, as so often happens, the question is not well posed. The issue is what average is being asked for?

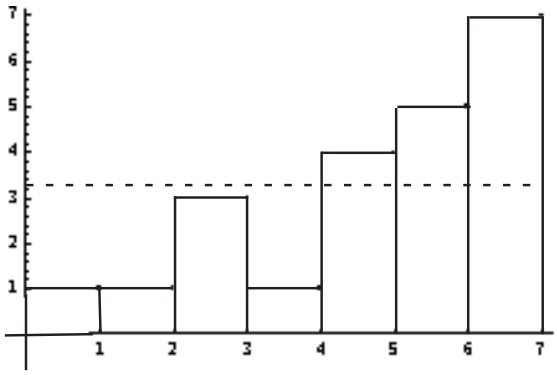

Figure 6.14: **Plot of Discrete Function for Averaging** The set of numbers, 1,1,3,1,4,5,7, are plotted as a discrete function in terms of the postion of the number in the table. In addition, a bar is drawn from the next lowest location at the height of the value. Also the average, $\frac{22}{7}$ is shown as a dotted line. The area under the barred segments and the area under the dotted line are the same. This allows a more general definition of the process of averaging: the average times the interval is equal to the area under the barred plot of the discrete function generated by the set of numbers.

How do you compute an average? What is the average of the set of numbers 1,1,3,1,4,5,7. The rule is that you add up all the numbers, the sum is 22, and divide by the number of numbers which is 7. The result is $\frac{22}{7}$ or a pretty good π. Looking at this process more closely, you realize that what we have is an ordered set of numbers: the first number is 1, the second

number is 1, the third number is 3, and so forth. We have a mapping of the set of integers onto our set of numbers, a discrete function. In this language, we can say that to compute the average by sequencing through our ordered set: add the first number to the second, add that sum to the third, add that sum to the fourth, and so forth. You divide by the number of times you take a number. We can display this algorithm for this case in the form

$$\text{Average} \equiv \frac{\sum_{i=1}^{n} f(i)}{n} \tag{6.23}$$

where $f(i)$ is the value of our discrete function for the i element of the table and n is the number of entries or more interesting as a plot of the discrete function that we have generated, see Figure 6.14 on page 179. In addition to plotting the function as a bar graph, the average is shown as a horizontal dotted line. From the figure, it can be seen that the area under the bars of the bar graph and the area under the dotted line are the the same. This leads to an alternative algorithm for finding the average of a set of numbers: construct the bar graph for the set of numbers and calculate the area under the bar graph divide this area by the number of elements in the set. The advantage of this definition is that it is easy to extend to situations where you want the average over a continuously varying set. An algorithm for this definition is:

$$\text{Average} \equiv \frac{\sum_{i=1}^{n} f(i)\Delta i}{\sum_{i=1}^{n} \Delta i} \tag{6.24}$$

where Δi is the width of the elements of the bar graph.

From this construction, the more general definition of the average can be developed that will work for continuous functions. The integral form of this same definition is

$$Average \equiv < f >_x \equiv \frac{\int_{x_0}^{x_f} f(x)dx}{\int_{x_0}^{x_f} dx} \tag{6.25}$$

where I have introduced a standard notation for taking the average. The subscript x indicates that the average is weighted by the variable x. The important point is that in different circumstances different weighting factors are appropriate and, although the definition looks as if it is independent of the choice of the weighting factor, it is not.

Now let's go back to our problem of the trip from Austin to College Station. To calculate an average, we need a set of numbers. How do we get the numbers? We have to decide what the weighting factor is. There are an infinity of choices but two are particularly obvious, time slicing and

space slicing. Were it not for a particular property of time slicing, space slicing is the easier because you will generally know how fast you can go at a given place. Thus to get the average velocity for space slicing choose spatial intervals and find the velocity in each. Applying this method to the Austin-College Station trip would yield the result that a speed of 75 $\frac{\text{m}}{\text{hr}}$ in the second segment would give an average speed of 50 $\frac{\text{m}}{\text{hr}}$.

The more accepted answer is the one that comes from using time slicing. In this case, the average is computed simply for a kinematic quantity like velocity because it is defined in terms of a time derivative. In other words,

$$\begin{aligned} < v >_t &= \frac{\int_{t_0}^{t_f} v(t)dt}{\int_{t_0}^{t_f} dt} \\ &= \frac{\int_{t_0}^{t_f} \frac{dx}{dt}dt}{t_f - t_0} \\ &= \frac{x_f - x_0}{t_f - t_0}, \end{aligned} \tag{6.26}$$

and thus the average velocity is just the net displacement divided by the total time interval. You loss track of the fact that you time sliced. Unless stated otherwise it is customary to assume that what is wanted is the time averaged case.

In Section 6.2.1 on page 166, there was some question regarding the height to use in the Lagrangian since it varied in the segment. We now see that the correct choice is the time average since the action is time sliced. For cases where you replace the curved trajectory with a straight line the two averages always come out the same and thus our substitution was correct. In cases where you are using a more subtle structure such as in Section 6.2.2 on page 169, you would get the wrong answer by substituting the mean position.

It also important to note that the action principle always uses time slicing – it is a part of the definition. It could turn out that, in some applications, a different slicing is easier to understand, see Section 15.1 on page 331. In fact, when we did Fermat least time, we did segment slicing. Whatever slicing technique is chosen, the action must always be evaluated using a time slicing.